Applications over Complex Lagrangians

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Abstract

In this paper, Lagrangian formalisms of Classical Mechanics was deduced on Kaehlerian manifold being geometric model of a generalized Lagrange space.Then, it was given two applications of complex Euler-Lagrange equations on mechanics system.

Key words: Complex and Kaehlerian manifold, Lagrangian systems, Maple.

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1. Introduction

As well known, modern differential geometry provides a suitable fields for studying Lagrangian theory of Classical Mechanics. This is easily shown by numerous articles and books [1, 2, 3, 4, 5, 6] and there in. Therefore the dynamics of a Lagrangian system is determined by a suitable vector field X defined on the tangent bundle of a given configuration space-manifold. If one takes an configuration manifold M and a regular Lagrangian function L on tangent bundle TM then it is seen that there is an unique vector field X on TM such that

$$i_X\omega_L = dE_L \tag{1}$$

where ω_L is the symplectic form and E_L is energy associated to L. The vector field X is a second order differential equation(semispray) since its integral curves are the solutions of the Euler-Lagrangian equations. Here, we present the complex Euler-Lagrange equations on Kaehlerian manifold being geometric model of a generalized Lagrange space and to derive complex Euler-Lagrange equations on two physical problems using Maple[7]. Hereafter, all mappings and manifolds are assumed to be differentiable of class C^∞ and the sum is taken over repeated indices. Also, we denote by $\mathcal{F}(TM)$ the set of complex functions on TM, by $\chi(TM)$ the set of complex vector fields on TM and by $\wedge^1(TM)$ the set of complex 1-forms on TM. $1 \le i \le n$.

2. Complex and Kaehlerian Manifolds

Let M configuration manifold. A tensor field J on TM is called an *almost complex structure* on TM if at every point p of TM, J is endomorphism of the tangent space $T_p(TM)$ such that $\mathbf{J}^2 = -\mathbf{I}$. A manifold TM with fixed almost complex structure J is called *almost complex manifold*. If (x^i) and $(x^i,\, y^i)$ are coordinate systems of M and TM, then $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ and $\{dx^i, dy^i\}$ are natural bases over $\mathbf{R}$ of the tangent space $T_p(TM)$ and the cotangent space $T_p^*(TM)$ of TM, respectively. Thus we get

$$J(\tfrac{\partial}{\partial x^i}) = \tfrac{\partial}{\partial y^i}, J(\tfrac{\partial}{\partial y^i}) = -\tfrac{\partial}{\partial x^i}. \tag{2}$$

Let $z^i = x^i + \mathbf{i}\, y^i$, $\mathbf{i} = \sqrt{-1}$, be a complex local coordinate system of TM. We define

$$\tfrac{\partial}{\partial z^i} = \tfrac{1}{2}\{\tfrac{\partial}{\partial x^i} - \mathbf{i}\tfrac{\partial}{\partial y^i}\},\ \tfrac{\partial}{\partial \overline{z}^i} = \tfrac{1}{2}\{\tfrac{\partial}{\partial x^i} + \mathbf{i}\tfrac{\partial}{\partial y^i}\},\ dz^i = dx^i + \mathbf{i}dy^i,\ d\overline{z}^i = dx^i - \mathbf{i}dy^i, \tag{3}$$

where $\frac{\partial}{\partial z^i}$ and dz^irepresent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then we calculate

$$J(\tfrac{\partial}{\partial z^i}) = \mathbf{i}\tfrac{\partial}{\partial z^i}, J(\tfrac{\partial}{\partial \overline{z}^i}) = -\mathbf{i}\tfrac{\partial}{\partial \overline{z}^i}. \tag{4}$$

Hermitian metric on an almost complex manifold with almost complex structure J is a Riemannian metric g on TM such that

$$g(JX, Y) + g(X, JY) = 0, \quad \forall X, Y \in \chi(TM). \tag{5}$$

An almost complex manifold TM with a Hermitian metric is called an *almost Hermitian manifold*. If TM is a complex manifold, then TM is called a *Hermitian manifold*. Let further TM be a 2m-dimensional almost Hermitian manifold with almost complex structure J and Hermitian metric g. The triple (TM, J, g) is called an *almost Hermitian structure*. Let (TM, J, g) be an almost Hermitian structure. The 2-form defined by

$$\Phi(X,Y) = g(X, JY), \quad \forall X, Y \in \chi(TM) \tag{6}$$

is called *the Kaehlerian form* of (TM, J, g). An almost Hermitian manifold is called *almost Kaehlerian* if its Kaehlerian form Φ is closed. If, moreover, TM is Hermitian, then TM is called a Kaehlerian manifold.

3. Complex Euler-Lagrange Equations

In this section, we deduce complex Euler-Lagrange equations for Classical Mechanics structured on Kaehlerian manifold. Let J be an almost complex structure on the Kaehlerian manifold and $(z^i, \overline{z}^i)$ its complex coordinates. The semispray ξ and Liouville vector field $V = J\xi$ on the Kaehlerian manifold are given by

$$\xi = \xi^i \tfrac{\partial}{\partial z^i} + \overline{\xi}^i \tfrac{\partial}{\partial \overline{z}^i}, \;\; J\xi = \mathbf{i}\xi^i \tfrac{\partial}{\partial z^i} - \mathbf{i}\overline{\xi}^i \tfrac{\partial}{\partial \overline{z}^i}. \tag{7}$$

We call *the kinetic energy* and *the potential energy of system* the maps given by $T, P : TM \to \mathbf{C}$. Then *Lagrangian energy function* L is the map $L : TM \to \mathbf{C}$ such that $L = T - P$ and *the energy function* E_L associated L is the function given by $E_L = V(L) - L$. The closed Kaehlerian form Φ_L is the closed 2-form given by $\Phi_L = -dd_J L$ such that $d_J = \mathbf{i}\frac{\partial}{\partial z^i} dz^i - \mathbf{i}\frac{\partial}{\partial \overline{z}^i} d\overline{z}^i : \mathcal{F}(TM) \to \wedge^1 TM$. Then we have

$$\Phi_L = \;\; \mathbf{i}\tfrac{\partial^2 L}{\partial z^j \partial z^i} dz^i \wedge dz^j + \mathbf{i}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} dz^i \wedge dz^j + \mathbf{i}\tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} dz^j \wedge d\overline{z}^i + \mathbf{i}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j \wedge d\overline{z}^i. \tag{8}$$

Since the map $TM_{\Phi_L} : \chi(TM) \to \wedge^1(TM)$ such that $TM_{\Phi_L}(\xi) = i_\xi \Phi_L$ is an isomorphism, there exists an unique vector ξ on TM such that the vector field ξ holds the equality given by (1). Thus vector field ξ on TM is seen as a *Lagrangian vector field* associated energy L on Kaehlerian manifold TM. Then

$$\begin{aligned} i_\xi \Phi_L = \;\; & \mathbf{i}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i} dz^j - \mathbf{i}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i} \delta_i^j dz^i + \mathbf{i}\xi^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} d\overline{z}^j - \mathbf{i}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} \delta_i^j dz^i \\ & + \mathbf{i}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} d\overline{z}^i - \mathbf{i}\overline{\xi}^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} \delta_i^j dz^j + \mathbf{i}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} \delta_i^j d\overline{z}^i - \mathbf{i}\overline{\xi}^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} d\overline{z}^j. \end{aligned} \tag{9}$$

Since the closed Kaehlerian form Φ_L on TM is symplectic structure, we have

$$E_L = \mathbf{i}\xi^i \tfrac{\partial L}{\partial z^i} - \mathbf{i}\overline{\xi}^i \tfrac{\partial L}{\partial \overline{z}^i} - L, \tag{10}$$

and hence

$$dE_L = \mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i} dz^j - \frac{\partial L}{\partial z^j} dz^j + \mathbf{i}\xi^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i} d\overline{z}^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j - \frac{\partial L}{\partial \overline{z}^j} d\overline{z}^j. \tag{11}$$

Considering **Eq.**(1) and the integral curve $\alpha : \mathbf{C} \to TM$ of ξ,i.e. $\xi(\alpha(t)) = \frac{d\alpha(t)}{dt}$, hence it is satisfied equations

$$-\mathbf{i}\left[\xi^j \frac{\partial^2 L}{\partial z^j \partial z^i} + \overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}\right] dz^j + \frac{\partial L}{\partial z^j} dz^j + \mathbf{i}\left[\xi^j \frac{\partial^2 L}{\partial z^j \partial \dot{z}^j} + \overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \dot{z}^j}\right] d\overline{z}^j + \frac{\partial L}{\partial \dot{z}^j} d\overline{z}^j = 0, \tag{12}$$

where the dots mean derivatives with respect to the time. Then we have

$$\mathbf{i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}^i}\right) - \frac{\partial L}{\partial z^i} = 0, \ \mathbf{i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}^i}\right) + \frac{\partial L}{\partial \dot{z}^i} = 0, \tag{13}$$

These equations infer *complex Euler-Lagrange equations* whose solutions are the paths of the semispray ξ on Kaehlerian manifold TM. Then (TM, Φ_L, ξ) is a *complex Lagrangian system* on Kaehlerian manifold TM.

Application 1: [3]Let us consider the system illustrated in **Figure1**. It consists of a light rigid rod of length ℓ, carrying a mass m at one end, and hinged at the other end to a vertical axis, so that it can swing freely in a vertical plane and accelerate along the vertical axis. Let us obtain the equations of motion for small oscillations by writing complex Lagrange function. Complex Lagrangian function of the system is

$$L = \frac{1}{2}m\left(\frac{\ell^2(\dot{z}+\dot{\overline{z}})^2}{A} + \frac{B^2}{4A} - \frac{1}{4}(\dot{z} - \dot{\overline{z}})^2 + \frac{\ell\sin\theta(\dot{z}+\dot{\overline{z}})[\mathbf{i}(\dot{z}-\dot{\overline{z}}) - \frac{B}{A^{1/2}}]}{A^{1/2}}\right) - \frac{1}{2}\mathbf{i}mg(z - \overline{z})$$

where $A = 4\ell^2 - (z + \overline{z})^2$, $B = (z + \overline{z})(\dot{z} + \dot{\overline{z}})$. Then, considering **Eq.**(13), the complex-Lagrangian equations of the motion on the mechanical system, can be calculated by

$$\mathbf{i}\frac{d}{dt}Q - Q = 0, \ \mathbf{i}\frac{d}{dt}W + W = 0,$$

such that

$$\begin{aligned} Q = & \frac{\ell^2(\dot{z}+\dot{\overline{z}})B}{A^2} + \frac{(\dot{z}+\dot{\overline{z}})B}{4A} + \frac{(z+\overline{z})B^2}{4A^2} + \frac{\ell B\sin\theta)[\mathbf{i}(\dot{z}-\dot{\overline{z}}) - \frac{B}{A^{1/2}}]}{2A^{3/2}} \\ & - \frac{\ell B\sin\theta(\dot{z}+\dot{\overline{z}})[\dot{z}+\dot{\overline{z}}+\frac{(z+\overline{z})B}{A}]}{2A} - \frac{1}{2}\mathbf{i}g \end{aligned}$$

and

$$W = \frac{\ell^2(\dot{z}+\dot{\overline{z}})}{A} + \frac{(\dot{z}+\dot{\overline{z}})B}{4A} - \frac{1}{4}\,\dot{z} + \frac{1}{4}\,\dot{\overline{z}} + \frac{\ell\sin\theta[\mathbf{i}(\dot{z}-\dot{\overline{z}}) - \frac{B}{A^{1/2}}]}{2A^{1/2}} - \frac{\ell B\sin\theta(\dot{z}+\dot{\overline{z}})[\mathbf{i} - \frac{(z+\overline{z})}{A^{1/2}}]}{2A^{1/2}}.$$

Application 2: [6]Let us consider the system illustrated in **Figure2**. A central force field $f(\rho) = A\rho^{\alpha-1}(\alpha \neq 0, 1)$ acts on a body with mass m in a constant gravitational field. Then let us find out the Euler-Lagrange equations of the motion by assuming the body always on the vertical plane.

The Lagrangian function of the system is,

$$L = \frac{1}{2}m\,\dot{z}\dot{\overline{z}} - \frac{A}{\alpha}(\sqrt{z\overline{z}})^{\alpha} - \mathbf{j}mg\frac{(z-\overline{z})\sqrt{z\overline{z}}}{(z+\overline{z})\sqrt{1-\frac{(z-\overline{z})^2}{(z+\overline{z})^2}}}.$$

Then, using **Eq.**(13), the so-called Euler-Lagrange equations of the motion on the mechanical systems, can be obtained, as follows:

$$L1: \quad \mathbf{i}\frac{\partial}{\partial t}S - S = 0, \qquad L2: \quad \mathbf{i}\frac{\partial}{\partial t}U + U = 0,$$

such that

$$\begin{aligned} S \;=\; & -\frac{A}{2z}(\sqrt{z\overline{z}})^{\alpha} + \mathbf{i}\frac{mg(z-\overline{z})\overline{z}}{2\sqrt{z\overline{z}}(z+\overline{z})W} + \mathbf{i}\frac{mg\sqrt{z\overline{z}}}{(z+\overline{z})W} \\ & -\mathbf{i}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})}{(z+\overline{z})^2W} - \mathbf{i}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})(-\frac{(z-\overline{z})}{(z+\overline{z})^2}+\frac{(z-\overline{z})^2}{(z+\overline{z})^3})}{(z+\overline{z})W^3}, \end{aligned}$$

and

$$U = \frac{1}{2}m\,\dot{\overline{z}}$$

where $W = \sqrt{1-\frac{(z-\overline{z})^2}{(z+\overline{z})^2}}$.

Conclusion: In this study, the Lagrangian formalisms and systems in Classical Mechanics had been intrinsically obtained making two complex applications.

References

[1] De Leon M., Rodrigues P.R., Generalized Classical Mechanics and Field Theory, North-Holland Math. Stud., 112, Elsevier Sci. Pub. Comp., Inc., New York, 1985.

[2] Norbury, J.W., Lagrangians and Hamiltonians for High School Students, arXiv: physics/0004029v1, 2000.

[3] Cabar G., Hamiltonian systems, Master Thesis, Pamukkale University,Turkey, 2006.

[4] Gibbons G. W., Part III: Applications of Differential Geometry to Physics, Cambridge CB3 0WA, UK., 2006.

[5] Tekkoyun M., Görgülü A., Higher Order Complex Lagrangian and Hamiltonian Mechanics Systems" , Physics Letters A, vol.357, 261-269, 2006.

[6] Tekkoyun M., Cabar G., Complex Lagrangians and Hamiltonians, Journal of Arts and Sciences, Çankaya Üniv., Fen-Ed.Fak., Issue 8/December 2007 .

[7] Celik B., Maple and Mathematics with Maple, Nobel publication and distribution, Turkey, 2004.